LECTURE 1: MOTIVIC COHOMOLOGY

1. Introduction

Let F be a number field, with ring of integers $\mathcal{O}_F$. Let $d = [F : \mathbb{Q}]$, so $d = r_1 + 2r_2$ where r_1 is the number of real embeddings $F \hookrightarrow \mathbb{R}$ and r_2 is the number of conjugate pairs of complex embeddings $F \hookrightarrow \mathbb{C}$. Write $\sigma_1, \dots, \sigma_{r_1} : F \hookrightarrow \mathbb{R}$ for the real embeddings, and $\sigma_{r_1+1}, \dots, \sigma_{r_1+r_2}, \overline{\sigma}_{r_1+1}, \dots, \overline{\sigma}_{r_1+r_2} : F \hookrightarrow \mathbb{C}$ for the complex embeddings. I want to recall two facts from algebraic number theory. The first is the Dirichlet unit theorem:

Theorem 1.1 (Dirichlet unit theorem). *The unit group $\mathcal{O}_F^*$ is finitely generated. More precisely,*

$$\mathcal{O}_F^* \cong \mu(F) \oplus \mathbb{Z}^{r_1+r_2-1}$$

where $\mu(F)$ denotes the (finite) group of roots of unity contained in F.

How might one prove this? Well, you can consider the Dirichlet regulator (map)

$$\begin{aligned} r_F &: F^* \to \mathbb{R}^{r_1+r_2} \\ &\alpha \mapsto (\log|\sigma_1(\alpha)|, \dots, \log|\sigma_{r_1}(\alpha)|, \log|\sigma_{r_1}(\alpha)|^2, \dots, \log|\sigma_{r_1+r_2}(\alpha)|^2) \end{aligned}$$

and look at the image of $\mathcal{O}_F^*$. The first thing to notice is that $r_F(\mathcal{O}_F^*)$ lies in the hyperplane $\{(x_1, \dots, x_{r_1+r_2}) \in \mathbb{R}^{r_1+r_2}, | \sum x_i = 0\} \cong \mathbb{R}^{r_1+r_2-1}$, because $\prod_i \sigma_i(\alpha) \in \mathbb{Z}^* = \{\pm 1\}$ for $\alpha \in \mathcal{O}_F^*$. It is easy enough to see that $r_F(\mathcal{O}_F^*)$ is a lattice in $\mathbb{R}^{r_1+r_2-1}$ (i.e. a discrete subgroup, so $r_F(\mathcal{O}_F^*) \cong \mathbb{Z}^r$ for some r) and that the kernel of r_F is $\mu(F)$. The harder part is to check that the lattice has full rank, i.e $r = r_1 + r_2 - 1$.

The next fact I want to recall is the analytic class number formula. Let

$$\zeta_F(s) := \sum_{0 \neq I \subseteq \mathcal{O}_F} \mathrm{Nm}(I)^{-s}$$

be the Dedekind zeta function of F. Here the sum runs over all non-zero ideals of $\mathcal{O}_F$. Note that $\zeta_{\mathbb{Q}}(s) = \zeta(s)$ is the Riemann zeta function. It converges absolutely for $\mathrm{Re}(s) > 1$, has an analytic continuation to a meromorphic function on $\mathbb{C}$ with a single simple pole at $s = 1$, and satisfies a functional equation relating $s \leftrightsquigarrow 1 - s$. This is all due to Dirichlet, Dedekind, Hecke,... .

Theorem 1.2 (Analytic class number formula). *The residue of $\zeta_F(s)$ at $s = 1$ is*

$$\lim_{s \to 1} (s-1)\zeta_F(s) = \frac{2^{r_1+r_2}\pi^{r_2}|\mathrm{Cl}(F)|}{|\mathrm{Disc}(F)|^{\frac{1}{2}}|\mu(F)|} \cdot \mathrm{Reg}(F)$$

where:

- $\mathrm{Disc}(F)$ *is the discriminant of F, i.e.* $\det(\sigma_i(\beta_j))^2$ *for $\beta_1, \dots, \beta_d$ a $\mathbb{Z}$-basis of $\mathcal{O}_F \cong \mathbb{Z}^d$.*

- $\mathrm{Cl}(F)$ *is the class group of* F, *i.e. the group of fractional ideals of* $\mathcal{O}_F$ *modulo the principal fractional ideals.*
- $\mathrm{Reg}(F)$ *is the regulator of* F, *i.e. the covolume of the Dirichlet regulator* r_F, *i.e. any* $(r_1 + r_2 - 1) \times (r_1 + r_2 - 1)$*-minor of the matrix*

$$\begin{pmatrix} r_F(\alpha_1) \\ r_F(\alpha_2) \\ \vdots \\ r_F(\alpha_{r_1+r_2-1}) \end{pmatrix}$$

where $\alpha_1, \ldots, \alpha_{r_1+r_2-1}$ *is a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^{r_1+r_2-1} \subset \mathcal{O}_F^*$.

Remark 1.3. Note that under the functional equation, the analytic class number formula becomes

$$\lim_{s\to 0} s^{-(r_1+r_2-1)}\zeta_F(s) = -\frac{|\mathrm{Cl}(F)|}{|\mu(F)|} \cdot \mathrm{Reg}(F)\,.$$

2. Where we are going:

There exist other formulae of this type throughout number theory/arithmetic geometry, i.e. of the form

$$\text{residue of an } L\text{-function} \sim_{\mathbb{Q}^*} \text{regulator}$$

where the symbol $\sim_{\mathbb{Q}^*}$ means "equals to, up to multiplying by a non-zero rational number". For example, the Birch and Swinnerton-Dyer conjecture says

$$\lim_{s\to 1} (s-1)^{-\mathrm{rank} E(F)} L(E,s) \sim_{\mathbb{Q}^*} Reg(E)$$

for an elliptic curve E over a number field F. We will see how how all of these conjectures are particular instances of Beilinson's conjectures relating the residues of L-functions to (co)volumes of regulator maps on lattices defined by motivic cohomology groups into Deligne cohomology. For example, the Dirichlet regulator r_F will appears as

$$\begin{array}{ccc} H^1_{\mathcal{M}}(F,1) & \longrightarrow & H^1_{\mathcal{D}}(F,1) \\ \downarrow \cong & & \downarrow \cong \\ F^* & \xrightarrow{\ r_F\ } & \mathbb{R}^{r_1+r_2} \end{array}$$

Remarks 2.1.

(1) If it wasn't clear already, the term "regulator" is used interchangeably both for a map and for the covolume of a lattice under that map.
(2) The non-zero rational number obscured by the $\sim_{\mathbb{Q}^*}$ symbol is also interesting (arguably more so!); just look at the analytic class number formula. This factor is pinned down in the Tamagawa number conjecture of Bloch-Kato.
(3) Beilinson was an undergraduate student, I believe, when he made his conjectures!

There are now many approaches to motivic cohomology (none when Beilinson made his conjecture; that was part of it!). In this course we will take the simplest: Higher Chow groups. For more details see Owen Patashnick's course in February/March.

3. Higher Chow groups

The idea comes from the definition of singular homology in topology. Recall the standard n-simplex

$$\Delta^n := \left\{(x_i) \in \mathbb{R}^{n+1} \,|\, x_i \geq 0, \sum x_i = 1\right\}.$$

Let X be a topological space. Then a singular n-simplex is a continuous map

$$f : \Delta^n \to X.$$

Let $S_n(X) := \mathbb{Z}[\text{singular } n\text{-simplices}]$ be the free abelian group generated by singular n-simplices on X. Then there are face maps (restriction to the faces)

$$\partial_1, \ldots, \partial_n : S_n(X) \to S_{n-1}(X)$$

and we define $\partial := \sum_{i=0}^{n} (-1)^i \partial_i$. Then you can check that

$$S_\bullet(X) := \quad \ldots \xrightarrow{\partial} S_{n+1}(X) \xrightarrow{\partial} S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} \ldots$$

is a complex. Then the singular homology of X (with $\mathbb{Z}$-coefficients) is defined to be the homology of this complex:

$$H_n^{\text{sing}}(X, \mathbb{Z}) := H_n(S_\bullet(X)).$$

We want to mimic this construction in the algebraic category (i.e. for varieties over a field k, or schemes of finite type if you like, maybe separated at least). But we run into problems! First of all, Δ^n is not a variety. OK, so we need the algebraic version:

$$\Delta^n := \operatorname{Spec}\Big(k[t_0, \ldots, t_n]/(\sum t_i - 1)\Big) = V(t_1 + \ldots + t_n - 1) \subset \mathbb{A}_k^{n+1}.$$

That's fine, but now the problem is that defining n-simplices to be morphisms $f : \Delta^n \to X$ isn't going to work because there are far too few (morphisms in algebraic geometry are far more rigid objects than continuous maps in topology). What to do? Well, we can do quite a standard and useful thing to do in algebraic geometry: identify a morphism $f : \Delta^n \to X$ with its graph Γ_f in $X \times \Delta^n$:

$$\Gamma_f := \{(f(x), x) \,|\, x \in \Delta^n\} \subset X \times \Delta^n.$$

Then Γ_f is a closed subvariety of $X \times \Delta^n$. In general, write

$$Z^p(X \times \Delta^n) := \mathbb{Z}[\text{irreducible closed subvarieties of } X \times \Delta^n \text{ of codimension } p]$$

for the free abelian group generated by irreducible closed subvarieties of $X \times \Delta^n$ of codimension p. The elements of $Z^p(X \times \Delta^n)$ are called *(algebraic) cycles* on $X \times \Delta^n$, or *correspondences* from X to Δ^n.

Right, so we should take correspondences from X to Δ^n as our algebraic version of singular n-simplices. Ah! but we want face maps $\partial_i : Z^p(X, n) \to Z^p(X, n-1)$, but what happens if the cycle doesn't intersect a face properly? Well, let's just throw those bad cycles away and define

$$Z^p(X, n) := \{Z \in Z^p(X \times \Delta^n) \,|\, Z \text{ intersects the faces properly}\}.$$

(Here "intersects the faces properly" means that every irreducible component of Z meets all faces of $X \times \Delta^m$ in codimension p for all $m < n$. Some work is 'being swept under the rug here!) Then we do have restriction maps to each of the codimension 1 faces of $X \times \Delta^n$:

$$\partial_0, \ldots, \partial_n : Z^p(X,n) \to Z^p(X,n-1).$$

Just like before put $\partial := \sum_{i=0}^{n}(-1)^i \partial_i$ so that we get a complex of abelian groups

$$Z^p(X,\bullet) := \quad \ldots \xrightarrow{\partial} Z^p(X,n+1) \xrightarrow{\partial} Z^p(X,n) \xrightarrow{\partial} Z^p(X,n-1) \xrightarrow{\partial} \ldots.$$

Definition 3.1. The higher Chow groups of X are defined to be the homology of the above complex:

$$\mathrm{CH}^p(X,n) := H_n(Z^p(X,\bullet)).$$

We define the motivic cohomology of X as

$$H^i_{\mathcal{M}}(X,n) := \mathrm{CH}^n(X,2n-i).$$

Remark 3.2. There are other ways of defining motivic cohomology. The approach we have taken (Bloch's approach) is probably the most elementary and most amenable to computation. All definitions are known to agree when X is smooth.

Remark 3.3. The change of indices when moving from motivic cohomology notation and higher Chow group notation is very tricky, at least for me. I always have to think about it. One check is that is that $H^{2n}_{\mathcal{M}}(X,n)$ (sometimes called the *Chow diagonal*, see below for the reason) should be $\mathrm{CH}^n(X,0)$, and that $H^n_{\mathcal{M}}(X,n)$ (sometimes called the *Milnor diagonal*, see below for the reason) should be $\mathrm{CH}^n(X,n)$.

4. Basic properties of motivic cohomology

(1) Let $f : X \to Y$ be a proper morphism. Then there is a pushforward map

$$f_* : \mathrm{CH}^p(X,n) \to \mathrm{CH}^{p-d}(Y,n)$$

where $d = \dim X - \dim Y$. So motivic cohomology if covariant for proper morphisms.

(2) Let $f : X \to Y$ be a flat morphism. Then there is a pullback map

$$f^* : \mathrm{CH}^p(Y,n) \to \mathrm{CH}^p(X,n).$$

So motivic cohomology is contravariant for flat morphisms.

(2.5) In fact, if $f : X \to Y$ is a morphism of quasi-projective varieties with Y smooth then there is a pullback map

$$f^* : \mathrm{CH}^p(Y,n) \to \mathrm{CH}^p(X,n).$$

(3) Motivic cohomology is "$\mathbb{A}^1$-homotopy invariant", i.e.

$$\mathrm{CH}^p(X \times \mathbb{A}^m_k, n) \cong \mathrm{CH}^p(X,n)$$

(via $\mathrm{pr}_1^* : X \times \mathbb{A}^m_k \to X$.) For example, $\mathrm{CH}^p(k[T_1,\ldots,T_m],n) \cong \mathrm{CH}^p(k,n)$.

(4) Let X, Y be smooth quasi-projective varieties (for simplicity). Then there is a product map

$$\mathrm{CH}^p(X,n) \otimes \mathrm{CH}^q(X,m) \to \mathrm{CH}^{p+q}(X \times Y, n+m)$$

making $\mathrm{CH}^*(X,*)$ into a bi-graded ring. The product is given by pulling back cycles to $X \times Y$ and intersecting.

(5) Let $f : X \to Y$ be a proper morphism of smooth varieties. Then there is a projection formula:

$$f_*(Z_2 \cdot f^*(Z_1)) = f_*(Z_2) \cdot Z_1$$

for $Z_1 \in \mathrm{CH}^*(Y,*)$, $Z_2 \in \mathrm{CH}^*(X,*)$.

(6) Let $j : Z \hookrightarrow X$ be a closed subvariety of pure codimension $d = \dim X - \dim Z$. Let $i : U := X - Z \hookrightarrow X$ be the complement. Then the following sequence is exact:

$$\cdots \to \mathrm{CH}^p(Z,n) \xrightarrow{j_*} \mathrm{CH}^{p+r}(X,n) \xrightarrow{i^*} \mathrm{CH}^{p+r}(U,n) \to \mathrm{CH}^p(Z,n-1) \to \cdots$$

This sequence is called the localisation sequence. Or Mayer-Vietoris, or excision.

Remark 4.1. In the setting of (6), part of the motivation (one reason among many) for higher Chow groups was precisely the question of extending the localisation sequence for classical Chow groups

$$\mathrm{CH}^p(Z) \xrightarrow{j_*} \mathrm{CH}^{p+r}(X) \xrightarrow{i^*} \mathrm{CH}^{p+r}(U) \to 0$$

to the left.

Examples 4.2. Let us give some examples of motivic cohomology groups to convince you that you have seen some of them before.

(1) $\mathrm{CH}^p(X,0) \simeq \mathrm{CH}^p(X) := Z^p(X,0)/\sim_{\mathrm{rat}}$ is the Chow group of codimension p cycles on X. Here $\sim_{\mathrm{rat}}$ means rational equivalence (see the exercises). This is why the diagonal $H^{2p}_{\mathcal{M}}(X,p) = \mathrm{CH}^p(X,0) \simeq \mathrm{CH}^p(X)$ is sometimes called the Chow diagonal.

(1.5) As a special case of (1), we find that

$$H^2_{\mathcal{M}}(X,1) = \mathrm{CH}^1(X,0) \simeq \mathrm{CH}^1(X) \simeq \mathrm{Pic}(X)$$

is the Picard group of X (i.e. the group of line bundles on X up to isomorphism).

(1.5+ϵ) As a special special case of (1), if $X = \mathrm{Spec}\, R$ for R a Dedekind domain (e.g. $R = \mathcal{O}_F$ for a number field F), then

$$H^2_{\mathcal{M}}(R,1) = \mathrm{CH}^1(R,0) \simeq \mathrm{CH}^1(R) \simeq \mathrm{Pic}(R) \simeq \mathrm{Cl}(R)$$

is the class group of R.

(2)

$$H^i_{\mathcal{M}}(X,1) = \mathrm{CH}^1(X,2-i) \simeq \begin{cases} \mathcal{O}^*_X(X) & \text{if } i = 1 \\ \mathrm{Pic}(X) & \text{if } i = 2 \\ 0 & \text{if } i \neq 1,2\,. \end{cases}$$

(2.5) As a special case of (2), if $X = \mathrm{Spec}\, R$ for a ring R, then

$$\mathrm{CH}^1(R,1) \simeq R^*\,.$$

5. Lots of conjectures

In general, motivic cohomology is a very deep and interesting invariant of an algebraic variety (Cf. the rest of this course!). In some sense it is still in it's infancy. There are many conjectures which have deep connections to one another and to deep results in many areas of mathematics. Motivic cohomology is tied up with Grothendieck's idea of motives: they are the Ext-groups in the category of mixed motives (if such a thing exists!). It is also tied up with algebraic K-theory: there exists a spectral sequence

$$E_2^{p,q} = \mathrm{CH}^{-q}(X, -p-q) \Rightarrow K_{-p-q}(X)$$

which is a motivic analogue of the Atiyah-Hirzebruch spectral sequence in topology. It degenerates after tensoring with $\mathbb{Q}$, yielding the following generalisation to all K-groups of the Grothendieck-Riemann-Roch theorem:

$$K_n(X) \otimes \mathbb{Q} \simeq \bigoplus_p \mathrm{CH}^p(X, n) \otimes \mathbb{Q}.$$

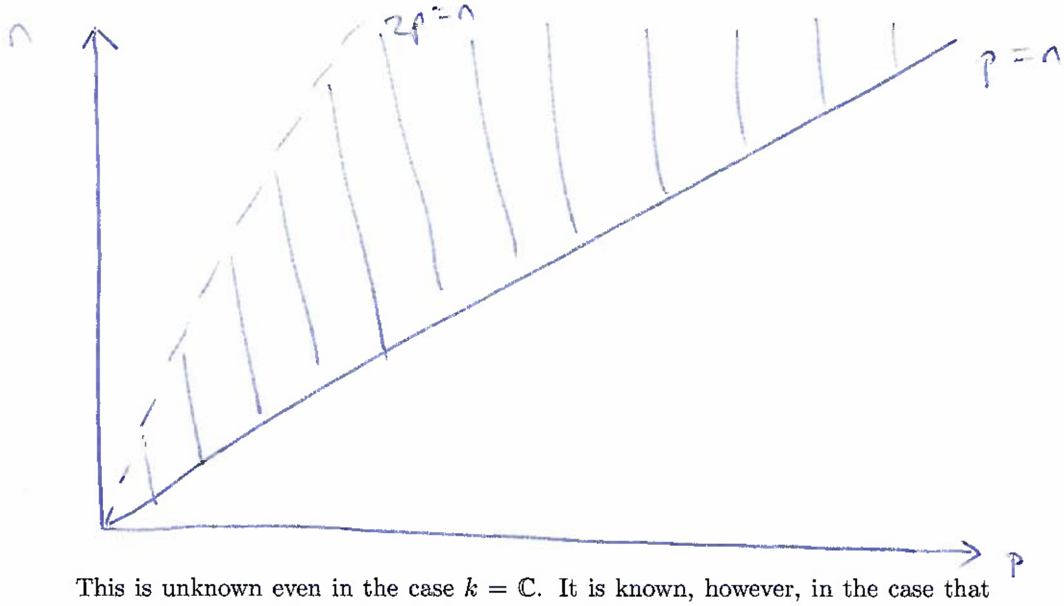
Here is a sample of some fundamental things that are still completely unknown. Each of which has important consequences (which we don't have time to go into, unfortunately):

- (Bass conjecture) Motivic cohomology groups are finitely generated.
- (Beilinson-Soulé conjecture) $H^i_{\mathcal{M}}(X, n) \otimes \mathbb{Q} = 0$ for $i < 0$.
- (Parshin conjecture) $H^i_{\mathcal{M}}(X, n) \otimes \mathbb{Q} = 0$ for $i \neq 2n$ (i.e. off of the Chow diagonal) for X smooth projective over a finite field.

Remark 5.1. Note that Bass+Parshin says that $H^i_{\mathcal{M}}(X, n)$ is a finite group away from the Chow diagonal, for smooth projective varieties X over a finite field.

6. Some calculations for fields

Let k be a field. Then the Beilinson-Soulé conjecture predicts that $\mathrm{CH}^p(k, n)$ can only be non-torsion in the following region:



This is unknown even in the case $k = \mathbb{C}$. It is known, however, in the case that $k = \mathbb{F}_q$ is a finite field, or $k = F$ is a number field:

- (Quillen):-

$$\mathrm{CH}^p(\mathbb{F}_q, n) \otimes \mathbb{Q} \simeq \begin{cases} 0 & \text{if } (p,n) \neq (0,0) \\ \mathbb{Q} & \text{if } (p,n) = (0,0). \end{cases}$$

- (Borel):-

$$H^i_{\mathcal{M}}(F,n)\otimes\mathbb{Q}\simeq\begin{cases}\mathbb{Q} & \text{if } i=n=0\\ F^*\otimes\mathbb{Q} & \text{if } i=n=1\\ \mathbb{Q}^{r_1+r_2} & \text{if } n>1, n \text{ even}, i=1\\ \mathbb{Q}^{r_2} & \text{if } n>0, n \text{ odd}, i=1\\ 0 & \text{else}\,.\end{cases}$$